Math 10A with Professor Stankova
Worksheet, Discussion #28; Monday, 10/30/2017
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Probability

Example

1. Suppose that the probability density function P that an atom emits a gamma wave satisfies the following differential equation $P' = -10P$ for $t \geq 0$ and $P(t) = 0$ for $t < 0$. Find P and calculate the CDF associated with P.

Solution: We solve the differential equation for P. We have that

$$\int \frac{dP}{P} = \int -10dt \implies \ln P = -10t + C \implies P = Ce^{-10t}.$$

Now in order for this to be a PDF, we need its integral to be 1. Thus, we have that

$$\int_{-\infty}^{\infty} P(t) = \int_{0}^{\infty} Ce^{-10t} = \frac{-Ce^{-10t}}{10}|_0^\infty = \frac{C}{10}.$$

Hence $C = 10$ and $P(t) = 10e^{-10t}$.

To find the CDF, we take the integral, for $t \geq 0$, we have that

$$F(t) = \int_{-\infty}^{t} 10e^{-10t} = \int_{0}^{t} 10e^{-10t} = -e^{-10t}|_0^t = 1 - e^{-10t}.$$

2. For the above PDF, find the probability that a gamma wave is emitted from -1 seconds to 1 second.

Solution: This is just the integral from -1 to 1. But, the probability is 0 from -1 to 0 so we just need to compute $P(0 \leq X \leq 1)$ which is

$$\int_0^1 P(t)dt = F(1) = 1 - e^{-10}.$$

Problems

3. True **FALSE** Since the CDF is an antiderivative of the PDF, there are multiple CDFs for a given PDF (and they differ by a $+C$).

Solution: We choose the CDF that gives us $F(-\infty) = 0$, which is like an initial condition and fixes the antiderivative.

4. True **FALSE** The area underneath a CDF must be equal to 1.

Solution: There is no such requirement on the CDF, and in fact, the area will actually diverge.

5. True **FALSE** A PDF must be continuous.

Solution: This is false. Look at the example above for a counterexample.

6. True **FALSE** Let $P(x) = Cx^3$ for $-1 \leq x \leq 2$ and 0 otherwise. Since $\int_{-1}^{3} P(x)dx = C(16 - 1/4)$, setting $C = (16 - 1/4)^{-1}$ makes P into a PDF.

Solution: P cannot possibly be a PDF because it is negative at some places.

7. Let $P(x) = Cx^2(10 - x)$ on $0 \leq x \leq 10$ and 0 otherwise. Find C such that P is a PDF and its corresponding CDF. Find the probability that we choose a number between 0 and 1.

Solution: In order for P to be a PDF, it needs to have integral 1. So

$$\int_{-\infty}^{\infty} P(x)dx = \int_{0}^{10} Cx^2(10 - x)dx = \frac{2500C}{3}.$$

Hence, we have that $C = \frac{3}{2500}$.
The CDF is the integral

$$\int_{-\infty}^{x} P(t)dt = \int_{0}^{x} \frac{3t^2(10 - t)}{2500} = \frac{(40 - 3x)x^3}{10000},$$

for $0 \leq x \leq 10$ and 0 for $x \leq 0$ and 1 for $x \geq 10$.

The probability that we choose a number between 0 and 1 is just the integral

$$\int_0^1 P(x)dx = \int_0^1 \frac{3x^2(10-x)}{2500}dx = \frac{37}{10000} = 0.37\%.$$

8. Let $P(x) = C(x-1)(x+1)$ on $-1 \leq x \leq 1$ and 0 otherwise. Find C such that P is a PDF and its corresponding CDF. Find the probability that we choose a number between 0 and 1.

Solution: We need

$$\int_{-\infty}^{\infty} P(x)dx = \int_{-1}^{1} C(x-1)(x+1)dx = \frac{-4}{3}C = 1.$$

Hence $C = \frac{-3}{4}$. The CDF is

$$F(x) = \int_{-\infty}^{x} P(t)dt = \int_{-1}^{x} -3/4(t-1)(t+1)dt = \frac{-x^3+3x+2}{4}.$$

for $-1 \leq x \leq 1$ and 0 for $x \leq -1$ and 1 for $x \geq 1$.

The probability that we choose a number between 0 and 1 is

$$F(1) - F(0) = \int_0^1 P(x)dx = \int_0^1 -3/4(x-1)(x+1)dx = \frac{1}{2}.$$

9. Let $P(x)$ satisfy $\frac{dP}{dx} = 2x$ for $0 \leq x \leq 1$ and $P(x) = 0$ otherwise. Find P such that it is a PDF and its corresponding CDF. Find the probability that we choose a number between 0 and 1.

Solution: We have that $P(x) = x^2 + C$ for $0 \leq x \leq 2$ and for this to be a PDF, we require that

$$\int_{-\infty}^{\infty} P(x)dx = \int_0^1 x^2 + Cdx = \frac{1}{3} + C = 1.$$

Hence $C = \frac{2}{3}$. The CDF is

$$F(x) = \int_{-\infty}^{x} P(t)dt = \frac{x(x^2+2)}{3}$$

for $0 \le x \le 1$ and 0 for $x \le 0$ and 1 for $x \ge 1$.

The probability that we choose a number between 0 and 1 is 1 because our PDF is only between 0 and 1.

10. Let $F(x) = \frac{x-1}{x+1}$ for $x \ge 1$ and 0 for $x \le 1$. Show that F is a CDF. Find the PDF associated with it and the probability that we choose a number between 1 and 2.

Solution: This is a CDF because it is continuous since $F(1) = 0$ and $\lim_{x\to\infty} F(x) = 1$ and F is non-decreasing. The PDF is

$$f(x) = \frac{d}{dx}F(x) = \frac{2}{(x+1)^2}$$

for $x \ge 1$ and 0 for $x \le 1$. The probability that we choose a number between 1 and 2 is

$$\int_1^2 f(x)dx = F(2) - F(1) = \frac{1}{3}.$$

11. Find numbers A, B such that $A\arctan(x) + B$ is a CDF and find the PDF associated with it. Find the probability that we choose a number between 0 and 1.

Solution: We know that $\arctan(x)$ is nondecreasing and all we need is for the range to be $(0,1)$. The original range is $(-\pi/2, \pi/2)$ and so letting $A = 1/\pi$ changes our range to $(-1/2, 1/2)$, and shifting up by $B = 1/2$ gives a range of $(0,1)$. Thus, we have that $A = \frac{1}{\pi}$ and $B = \frac{1}{2}$. The PDF is

$$f(x) = \frac{d}{dx}F(x) = \frac{1}{\pi + \pi x^2}.$$

The probability that we choose a number between 0 and 1 is

$$\int_0^1 f(x)dx = F(1) - F(0) = \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

12. Let $F(x) = \ln x$ for $1 \le x \le a$ and $F(x) = 0$ for $x \le 1$ and $F(x) = 1$ for $x \ge a$. Find a such that F is a continuous CDF and find the PDF associated with it. Find the probability that we choose a number between 1 and 2.

Solution: In order for this to be a CDF, we require need this to be nondecreasing so a is such that $\ln(a) = 1$ or $a = e$. The PDF is the derivative so $f(x) = \frac{1}{x}$ between 1 and e and equal to 0 otherwise. The probability that we choose a number between 1 and 2 is $\ln(2) - \ln 1 = \ln 2$.

Logistic Growth

Example

13. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Write this down as a differential equation. Solve for the population if the initial population is 100.

Solution: This is written as

$$\frac{dP}{dt} = rP(1 - P/K) = 10P(1 - P/1000).$$

The solution is given as

$$P(t) = \frac{K}{1 + Ce^{-rt}} = \frac{1000}{1 + Ce^{-10t}}.$$

If the initial population is 100, then we have that

$$P(0) = 100 = \frac{1000}{1 + Ce^{0}} = \frac{1000}{1 + C}.$$

Therefore, we have that $1 + C = 10$ and $C = 9$. So

$$P(t) = \frac{1000}{1 + 9e^{-10t}}.$$

Problems

14. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Assume that the population is harvested at a rate proportional to its population with a constant of proportionality of 5. Write this down as a differential equation. What is the fate of the population for different initial sizes?

Solution: We have that

$$\frac{dP}{dt} = 10P(1 - P/1000) - 5P = 5x - x^2/100 = x/100(500 - x).$$

Thus, for all populations $x > 0$, the population will go towards 500, which is stable. The population 0 is unstable.

15. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Assume that the population is harvested at a rate proportional to its population with a constant of proportionality of 10. Write this down as a differential equation. What is the fate of the population for different initial sizes?

Solution: We have that

$$\frac{dP}{dt} = 10P(1 - P/1000) - 10P = -x^2/100.$$

Thus, for all populations $x \geq 0$, the population will go towards 0, and 0 is semistable.

16. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Assume that the population is harvested at a rate proportional to its population with a constant of proportionality of 15. Write this down as a differential equation. What is the fate of the population for different initial sizes?

Solution: We have that

$$\frac{dP}{dt} = 10P(1 - P/1000) - 15P = -5x - x^2/100 = -x/100(500 + x).$$

Thus, for all populations $x \geq 0$, the population will go towards 0, which is stable. The population -500 (which doesn't make sense in this model) is unstable.

17. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Assume that 2100 individuals are killed every year. Write this down as a differential equation. What is the fate of the population for different initial sizes?

Solution: We have that

$$\frac{dP}{dt} = 10P(1 - P/1000) - 2100 = 10x - x^2/100 - 2100 = -1/100(x - 300)(x - 700).$$

Thus, for all populations $x > 300$, the population will go towards 700, which is stable. For populations < 300, the population will go to extinction and 300 is unstable.

18. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Assume that 2500 individuals are killed every year. Write this down as a differential equation. What is the fate of the population for different initial sizes?

Solution: We have that

$$\frac{dP}{dt} = 10P(1 - P/1000) - 2500 = 10x - x^2/100 - 2100 = -1/100(x - 500)^2.$$

Thus, for all populations $x \geq 500$, the population will go towards 500, which is semistable. For populations < 500, the population will go to extinction.

19. The rate of growth of a population is logistic with an intrinsic rate of growth of $r = 10$ and a carrying capacity of 1000. Assume that 2900 individuals are killed every year. Write this down as a differential equation. What is the fate of the population for different initial sizes?

Solution: We have that

$$\frac{dP}{dt} = 10P(1 - P/1000) - 2900 = 10x - x^2/100 - 2100 = -1/100(x - 500)^2 - 400.$$

Thus, for all populations, the population will go to extinction and there is no way for this population to survive.